

ON THE TRIVIAL EXTENSION OF EQUIVALENCE RELATIONS ON ANALYTIC SPACES

BY

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ABSTRACT. In this paper, we shall consider the problem: let X be a (reduced) analytic space and A a nowhere dense analytic set in X . And let R be a proper equivalence relation on A such that the quotient space A/R is an analytic space, and $\tilde{R}$ the trivial extension of R to X . Then, is $X/\tilde{R}$ an analytic space? To this, we have three sufficient conditions. Moreover, using this result we shall extend Satz 1 of H. Kerner [8].

1. Introduction. Let $(X, \mathcal{O}_X)$ be an analytic space and R an equivalence relation on X . Then the local ringed quotient space $(X/R, \mathcal{O}_{X/R})$ is defined and the problem, whether $(X/R, \mathcal{O}_{X/R})$ is an analytic space, is studied by H. Cartan, H. Holmann, B. Kaup and others.

In this paper, we shall consider the problem: let X be a (reduced) analytic space and A a nowhere dense analytic set in X . And let R be a proper equivalence relation on A such that the quotient space A/R is an analytic space, and $\tilde{R}$ the trivial extension of R to X . Then, is $X/\tilde{R}$ an analytic space? To this, we have

THEOREM. $X/\tilde{R}$ is an analytic space, if one of the following three statements is satisfied:

- (1) R is finite.
- (2) A is contractible in X and the canonical mapping $j: A/R \rightarrow X/\tilde{R}$ is quasi-finite.
- (3) A is contractible and retractable in X .

Next, using Theorem, (3), we shall extend Satz 1 of H. Kerner [8]: let X_k be a connected complex manifold, A_k a contractible and retractable analytic set in X_k and R_k a proper equivalence relation on A_k such that A_k/R_k is an analytic space and $\dim_a R_k(a) > 0$ for any $a \in A_k$ ($k = 1, 2$). Then, we have the following diagrams of analytic spaces:

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$$\begin{array}{ccc}
 X_k & \xrightarrow{\tilde{p}_k} & X_k/\tilde{R}_k \\
 i_k \uparrow & & \uparrow j_k \\
 A_k & \xrightarrow{p_k} & A_k/R_k
 \end{array}$$

Here $p_k: A_k \rightarrow A_k/R_k$, $\tilde{p}_k: X_k \rightarrow X_k/\tilde{R}_k$ are natural projections, $i_k: A_k \rightarrow X_k$ is the injection and $j_k: A_k/R_k \rightarrow X_k/\tilde{R}_k$ is the canonical mapping. Let $r_k: X_k \rightarrow A_k$ be the holomorphic retraction. Then, we have

THEOREM. Suppose that f.m.d. $r_2 \geq \dim A_1 + 2$. If $X_1/\tilde{R}_1$ and $X_2/\tilde{R}_2$ are analytically equivalent, then the above two diagrams are analytically equivalent.

H. Kerner has treated the case that $r_k: X_k \rightarrow A_k$ is a weakly negative vector bundle and $R_k(a) = A_k$ for any $a \in A_k$.

2. Trivial extension of equivalence relations. Let L be the category of local ringed spaces [6]: objects in L are local ringed spaces and morphisms in L are morphisms of local ringed spaces.

DEFINITION 1. A commutative diagram of morphisms in L :

$$\begin{array}{ccc}
 Z & \xrightarrow{b} & P \\
 s \uparrow & & \uparrow a \\
 X & \xrightarrow{r} & Y
 \end{array}$$

is called a pushout (and P is called the pushout for r and s), if for any object A and morphisms $u: Y \rightarrow A$, $v: Z \rightarrow A$ in L with $v \circ s = u \circ r$, there exists the unique morphism $p: P \rightarrow A$ such that $p \circ b = v$ and $p \circ a = u$.

Let $(X, {}_X\mathcal{O})$ be a (reduced) analytic space and R an equivalence relation on X . Then there exists the local ringed quotient space $(X/R, {}_X\mathcal{O}/R)$ and the natural projection $p: X \rightarrow X/R$ is a morphism of local ringed spaces, where X/R is the quotient topological space of X by R and ${}_X\mathcal{O}/R$, the structure sheaf on X/R , is defined as follows: for any open set $U \subset X/R$, $({}_X\mathcal{O}/R)(U) := \{f: U \rightarrow \mathbb{C}, f \circ p \in \Gamma(p^{-1}(U), {}_X\mathcal{O})\}$.

DEFINITION 2. An equivalence relation R on X is called proper if for any compact set $K \subset X$, the R -saturated set $R(K)$ (i.e. the union of all equivalence classes meeting K) is also compact.

This condition is equivalent that X/R is locally compact and the natural projection $p: X \rightarrow X/R$ is proper.

DEFINITION 3. Let A be a subset of X and R an equivalence relation on A . The trivial extension $\tilde{R}$ of R to X , an equivalence relation on X , is defined by

$$\tilde{R}(x) := \begin{cases} R(x), & \text{for } x \in A, \\ \{x\}, & \text{for } x \notin A, \end{cases}$$

where $R(x)$, $x \in A$, denotes the equivalence class by R containing x .

Let $(A, {}_A 0)$ be a nowhere dense analytic set of $(X, {}_X 0)$ and R an equivalence relation on A . Then we have the local ringed quotient spaces $(A/R, {}_A 0/R)$, $(X/\tilde{R}, {}_X 0/\tilde{R})$. Let $p: A \rightarrow A/R$, $\tilde{p}: X \rightarrow X/\tilde{R}$ be natural projections and $i: A \rightarrow X$ the injection. Then there exists the canonical mapping $j: A/R \rightarrow X/\tilde{R}$ ($\tilde{p} \circ i = j \circ p$) and j is a morphism in L .

LEMMA 1. $X/\tilde{R}$ is the pushout for i and p in L .

PROOF. For any object Z and morphisms $u: A/R \rightarrow Z$, $v: X \rightarrow Z$ in L with $v \circ i = u \circ p$, we define the mapping as follows: for any $\tilde{x} \in X/\tilde{R}$, we put $\varphi(\tilde{x}) := v(x)$ ($x \in \tilde{p}^{-1}(\tilde{x})$). Then this is well defined. In fact $\tilde{p}(x) = \tilde{p}(x')$ ($x, x' \in X$) implies $v(x) = v(x')$. Now φ is continuous with $v = \varphi \circ \tilde{p}$, and $u = \varphi \circ j$ since $u \circ p = \varphi \circ j \circ p$ and p is surjective.

For any $f \in {}_Z 0_{\varphi(\tilde{x})}(\tilde{x} \in X/\tilde{R})$, there exists $\tilde{f} \in ({}_X 0/\tilde{R})_{\tilde{x}}$ with $v_x^*(f) = \tilde{f} \circ \tilde{p}$. And we put $\varphi_x^*(f) := \tilde{f}$. Then φ^* holds commutativity and

is unique. Hence $X/\tilde{R}$ is the pushout in L for i and p . Q.E.D.

DEFINITION 4. An analytic set $A \subset X$ is called contractible in X if A is nowhere discrete, compact and if there exist an analytic space Y and a surjective proper holomorphic mapping $\psi: X \rightarrow Y$ such that $\psi(A) =: y_A \in Y$ and the restriction $\psi|_{(X-A)} \rightarrow (Y - \{y_A\})$ is biholomorphic.

DEFINITION 5. An analytic set $A \subset X$ is called retractable if there exists a holomorphic retraction of X to A (i.e. a surjective holomorphic mapping $r: X \rightarrow A$ with $r|_A = \text{id}_A$).

DEFINITION 6. A morphism $f: (X, {}_X 0) \rightarrow (Y, {}_Y 0)$ in L is called quasi-finite if for any $x \in X$, ${}_X 0_x / (f_x^*(M_{f(x)}))$ is a finite dimensional vector space over $\mathbb{C}$, where $M_{f(x)}$ is the maximal ideal of ${}_Y 0_{f(x)}$.

Let $(A, {}_A 0)$ be an analytic set in $(X, {}_X 0)$ and R a proper equivalence relation on A such that A/R is an analytic space. Using the results by B. Kaup [6] and the method of H. Kerner [8], we shall show the sufficient conditions under which $X/\tilde{R}$ is an analytic space.

THEOREM 1. $X/\tilde{R}$ is an analytic space, if one of the following statements is satisfied:

- (1) R is finite (i.e. every equivalence class of A by R is a finite set).
- (2) A is contractible in X and the canonical mapping $j: A/R \rightarrow X/\tilde{R}$ is quasi-finite.
- (3) A is contractible and retractable in X .

PROOF. (1) From Lemma 1, $X/\tilde{R}$ is the pushout for the injection $i: A \rightarrow X$ and the natural projection $p: A \rightarrow A/R$. Hence, by B. Kaup [6, Satz 1.8], $X/\tilde{R}$ is an analytic space.

(2) If A is contractible in X , A is exceptional in X in the sense of B. Kaup [6]. Hence, by Lemma 1 and B. Kaup [6, Aussage 1.11], $X/\tilde{R}$ is an analytic space.

(3) $\tilde{R}$ is proper since, for any compact set $K \subset X$, $\tilde{R}(K) = K \cup R(K)$ is also compact in X .

By the assumption, there exist an analytic space Y , a surjective proper holomorphic mapping $\psi: X \rightarrow Y$ and a holomorphic retraction $r: X \rightarrow A$. Then we have a surjective morphism $\tilde{r}: X/\tilde{R} \rightarrow A/R$ with $\tilde{r} \circ \tilde{p} = p \circ r$. In fact, for any $\tilde{x} \in X/\tilde{R}$, we put

$$\tilde{r}(\tilde{x}) := p \circ r(x) \quad (x \in \tilde{p}^{-1}(\tilde{x})).$$

Then $\tilde{r}: X/\tilde{R} \rightarrow A/R$ is well defined.

$$\begin{array}{ccc} X & \xrightarrow{\tilde{p}} & X/\tilde{R} \\ i \uparrow & & \downarrow \tilde{r} \\ A & \xrightarrow{p} & A/R \end{array}$$

Now, we claim that $(X/\tilde{R}, {}_X\mathcal{O}/\tilde{R})$ is locally morph-separable (i.e. for any $\tilde{x} \in X/\tilde{R}$, there exists an open neighborhood $U \subset X/\tilde{R}$ such that $\Gamma(U, {}_X\mathcal{O}/\tilde{R})$ separates points of U). Then $(X/\tilde{R}, {}_X\mathcal{O}/\tilde{R})$ is an analytic space by H. Cartan [1, Main Theorem].

Let $\tilde{x}$ be a point of $X/\tilde{R}$. We may assume that $\tilde{x} \in j(A/R)$. Then there exists an open neighborhood V of $x := \tilde{r}(\tilde{x})$ such that $\Gamma(V, {}_A\mathcal{O}/R)$ separates points of V and also there exists an open neighborhood $O \subset Y$ of y_A such that $\Gamma(O, {}_Y\mathcal{O})$ separates points of O . Since $W := \psi^{-1}(O) \subset X$ is an open neighborhood of A , we have $\tilde{p}^{-1}(\tilde{p}(W)) = W$, hence $\tilde{p}(W)$ is an open neighborhood of $\tilde{x}$. Thus, so is $U := \tilde{p}(W) \cap \tilde{r}^{-1}(V) \subset X/\tilde{R}$. We can show that U satisfies the above statement. Let $\tilde{y}, \tilde{z}$ be any distinct points in U . Then there exist two distinct points y, z in X such that $\tilde{p}(y) = \tilde{y}$, $\tilde{p}(z) = \tilde{z}$. If $\psi(y) \neq \psi(z)$, we have $f \in \Gamma(O, {}_Y\mathcal{O})$ with $f \circ \psi(y) \neq f \circ \psi(z)$. And $f \circ \psi \in \Gamma(W, {}_X\mathcal{O})$ is constant on A . Put

$$F(\tilde{w}) := \begin{cases} f \circ \psi \circ (\tilde{p}|W - A)^{-1}(\tilde{w}), & \text{for } \tilde{w} \in \tilde{p}(W - A), \\ f(y_A), & \text{for } \tilde{w} \in \tilde{p}(A). \end{cases}$$

Then $F \in \Gamma(\tilde{p}(W), {}_X 0/\tilde{R}) \subset \Gamma(U, {}_X 0/\tilde{R})$ and $f \circ \psi = F \circ \tilde{p}$ in W . Therefore $F(\tilde{y}) \neq F(\tilde{z})$. If $\psi(y) = \psi(z)$, then $y, z \in A$ and $p(y) \neq p(z)$. Hence we have $g \in \Gamma(V, {}_A 0/R)$ with $g \circ p(y) \neq g \circ p(z)$. Put in U , $G := g \circ \tilde{r}$; then $G \in \Gamma(U, {}_X 0/\tilde{R})$ with $G(\tilde{y}) \neq G(\tilde{z})$, since $r: X \rightarrow A$ is a holomorphic retraction. Thus $(X/\tilde{R}, {}_X 0/\tilde{R})$ is locally morph-separable. Q.E.D.

REMARK 1. We can easily find the examples such that $X/\tilde{R}$ is not an analytic space, in the case that R is not finite in (1), or A is not contractible in (2), (3) respectively.

COROLLARY 1. Let $(X, {}_X 0)$, $(A, {}_A 0)$ and R be as in Theorem 1, (1) or (3). Then A/R is embedded in $X/\tilde{R}$. In particular, in the case of (3), A/R is contractible and retractable in $X/\tilde{R}$.

PROOF. The canonical mapping $j: A/R \rightarrow j(A/R)$ is a holomorphic homeomorphism since j is proper. We assert that for any $\tilde{a} \in A/R$, $j_a^*: ({}_X 0/\tilde{R})_{j(\tilde{a})} \rightarrow ({}_A 0/R)_{\tilde{a}}$ is surjective.

(1) For any $f \in ({}_A 0/R)_{\tilde{a}}$ ($\tilde{a} \in A/R$), we have $p_a^*(f) \in {}_A 0_a$ ($a \in p^{-1}(\tilde{a})$). Then there exists $g \in {}_X 0_a$ with $i_a^*(g) = p_a^*(f)$. Since p is finite proper, we have $G \in ({}_X 0/\tilde{R})_{j(\tilde{a})}$ with $\tilde{p}_a^*(G) = g$. Then it follows that $j_a^*(G) = f$.

(3) Since $\tilde{r} \circ j = \text{id}_{A/R}$, surjectiveness of j_a^* is evident and in particular $\tilde{r}$ is a holomorphic retraction. Therefore A/R is retractable and contractible in $X/\tilde{R}$. Q.E.D.

3. Applications. We now consider the following problem: Let $(X, {}_X 0)$ and $(M, {}_M 0)$ be analytic spaces, A a nowhere dense analytic set in X and $h: A \rightarrow M$ a surjective proper holomorphic mapping. Then, does an analytic space Y exist with the following property (P)?

(P) There exist a surjective proper holomorphic mapping $\tilde{h}: X \rightarrow Y$ and an injection $j: M \rightarrow Y$ such that the restriction $\tilde{h}|A = j \circ h$ and $\tilde{h}|(X - A) \rightarrow (Y - \tilde{A})$ ($\tilde{A} := \tilde{h}(A)$) is biholomorphic.

DEFINITION 7. We say that a reduced analytic space X is maximal if, for any open set $U \subset X$ and a nowhere dense analytic set $S \subset U$, every continuous function on U which is holomorphic on $U - S$ is actually holomorphic on U .

REMARK 2. If an analytic space $(X, {}_X 0)$ is maximal, ${}_X 0$ is the maximal reduced complex structure on X .

Let X , A and R be as in Theorem 1 (1) or (2) or (3). If X is maximal, so is $X/\tilde{R}$.

Let R_h be the equivalence relation on A defined by $h: A \rightarrow M$ (i.e. for any $u, v \in A$, $u R_h v$ means $h(u) = h(v)$). Then R_h is proper and, if M is maximal we can show that $A/R_h, M$ are isomorphic. Thus from Theorem 1 and Corollary 1, we have

THEOREM 2. *If (1) or (3) in Theorem 1 is satisfied for X, A, R_h and M is maximal, there exists an analytic space Y with the property (P).*

COROLLARY 2. *Let X, A, M and R_h be as in Theorem 2. Suppose that X is maximal. Then any maximal analytic space Y' with the property (P) is biholomorphically equivalent to $X/\tilde{R}_h$.*

PROOF. Let $\tilde{p}': X \rightarrow Y'$ be a surjective proper holomorphic mapping and $j': A/R_h \rightarrow Y'$ an injection such that the restriction $\tilde{p}'|_A = j' \circ p$ and $\tilde{p}'|(X - A) \rightarrow (Y' - \tilde{p}'(A))$ is biholomorphic. Then, from Lemma 1, we have the unique holomorphic mapping $\psi: X/\tilde{R}_h \rightarrow Y'$ with $\tilde{p}' = \psi \circ \tilde{p}$, $j' = \psi \circ j$.

$$\begin{array}{ccccc}
 & & & & Y' \\
 & & & \nearrow \tilde{p}' & \\
 X & \xrightarrow{\tilde{p}} & X/\tilde{R}_h & \xrightarrow{\psi} & \\
 \uparrow i & & \uparrow j & \nearrow j' & \\
 A & \xrightarrow{p} & A/R_h & &
 \end{array}$$

Since the restriction $\psi|(X/\tilde{R}_h - \tilde{p}(A)) \rightarrow (Y' - \tilde{p}'(A))$ and $\psi|j(A/R_h) \rightarrow j'(A/R_h)$ are biholomorphic, ψ is bijective. Moreover, ψ^{-1} is continuous since $\tilde{p}'$ is proper. Hence ψ is a holomorphic homeomorphism. By assumption, Y' is maximal, thus ψ is biholomorphic. Q.E.D.

Now, using Theorem 1, (3), we shall extend Satz 1 of H. Kerner [8]. Let X_k be a connected complex manifold and A_k a contractible and retractable analytic set in X_k . Let R_k be an equivalence relation on A_k such that A_k/R_k is an analytic space and $\dim_a R_k(a) > 0$ for any $a \in A_k$ ($k = 1, 2$). If R_k is proper, $X_k/\tilde{R}_k$ is an analytic space and the natural projection $\tilde{p}_k: X_k \rightarrow X_k/\tilde{R}_k$ is proper holomorphic. Let $r_k: X_k \rightarrow A_k$ be the holomorphic retraction. Then we use the following result.

LEMMA 2 (H. HOLMANN [5]). *Let X be a complex manifold and A an analytic set in X . Suppose that $r: X \rightarrow A$ is a holomorphic retraction. Then A is a closed complex submanifold of X and, for any $a \in A$, there exists an open neighborhood $U \subset X$ such that the restriction $r|_U$ is a holomorphic projection (i.e. there exist two complex manifolds M_1, M_2 and a biholomorphic mapping $T: U \rightarrow M_1 \times M_2$ such that $\text{pr} = T \circ r \circ T^{-1}$, where $\text{pr}: M_1 \times M_2 \rightarrow M_1 \times$*

M_2 , $\text{pr}(x_1, x_2) = (x_1, x_2^0)$ for any $(x_1, x_2) \in M_1 \times M_2$, x_2^0 is a fixed point).

If φ is a holomorphic mapping of an analytic space X into an analytic space Y , we put $\text{f.m.d. } \varphi := \min_{x \in X} \dim_x \varphi^{-1}(\varphi(x))$. Then using Lemma 2 and the assumption $\dim_a R_k(a) > 0$, we can prove the next lemma in almost like manner as in [8].

LEMMA 3. Suppose that $\text{f.m.d. } r_k \geq 2$. Then $\tilde{A}_k := \tilde{p}_k(A_k)$ is the set of all singular points of $X_k/\tilde{R}_k$.

THEOREM 3. Suppose that $\text{f.m.d. } r_2 \geq \dim A_1 + 2$. If $X_1/\tilde{R}_1$ and $X_2/\tilde{R}_2$ are analytically equivalent, the following diagrams ($k = 1, 2$) are analytically equivalent.

PROOF. We first show that

$$(*) \quad \text{f.m.d. } r_1 \geq \dim A_2 + 2$$

in some open neighborhood of A_1 .

$$\begin{array}{ccc} X_k & \xrightarrow{\tilde{p}_k} & X_k/\tilde{R}_k \\ i_k \uparrow & & \uparrow j_k \\ A_k & \xrightarrow{p_k} & A_k/R_k \end{array}$$

By assumption, any point of A_k ($k = 1, 2$) has an open neighborhood with the property stated in Lemma 2. Let O_k be the union of all such open neighborhoods. Then

$$\dim O_2 - \dim A_2 \geq \text{f.m.d. } r_2 \geq \dim A_1 + 2.$$

Since $\dim O_1 = \dim O_2$, it follows that

$$\text{f.m.d. } (r_1|_{O_1}) = \dim O_1 - \dim A_1 \geq \dim A_2 + 2.$$

Hence, by Lemma 3, $\tilde{A}_k := \tilde{p}_k(A_k)$ ($k = 1, 2$) is the set of all singular points of $X_k/\tilde{R}_k$. Let $\psi: X_1/\tilde{R}_1 \rightarrow X_2/\tilde{R}_2$ be the biholomorphic mapping. Then $\psi(\tilde{A}_1) = \tilde{A}_2$ and there exists an open neighborhood $U_k \subset X_k/\tilde{R}_k$ of $\tilde{A}_k$ with $U_k^\wedge := \tilde{p}_k^{-1}(U_k) \subset O_k$.

We now assert that there exists a holomorphic mapping $\psi^\wedge: U_1^\wedge \rightarrow U_2^\wedge$ such that $\psi \circ \tilde{p}_1 = \tilde{p}_2 \circ \psi^\wedge$. We put

$$\psi^\sim := \psi|_{(U_1 - \tilde{A}_1)} \rightarrow (U_2 - \tilde{A}_2),$$

$$\tilde{p}_k^\sim := \tilde{p}_k|_{(U_k^\wedge - A_k)} \rightarrow (U_k - \tilde{A}_k) \quad (k = 1, 2).$$

These mappings are biholomorphic. And we put, on $U_1^\wedge - A_1$, $\tau := r_2 \circ (\tilde{p}_2^\sim)^{-1} \circ \psi^\sim \circ \tilde{p}_1^\sim$. Then $\tau: (U_1^\wedge - A_1) \rightarrow A_2$ is also holomorphic. Since f.m.d. $\tau \geq \dim A_1 + 2$ on $U_1^\wedge - A_1$, we have the holomorphic mapping $\tilde{\tau}: U_1^\wedge \rightarrow A_2$ such that $\tilde{\tau}|(U_1^\wedge - A_1) = \tau$ [9, Satz 2]. Define the mapping $\psi^\wedge: U_1^\wedge \rightarrow U_2^\wedge$ as follows:

$$\psi^\wedge(x) = \begin{cases} (\tilde{p}_2^\sim)^{-1} \circ \psi^\sim \circ \tilde{p}_1^\sim(x), & \text{for } x \in U_1^\wedge - A_1, \\ i_2 \circ \tilde{\tau}(x), & \text{for } x \in A_1, \end{cases}$$

where $i_2: A_2 \rightarrow U_2^\wedge$ is the injection. Remark that $\tilde{\tau} = r_2 \circ \psi^\wedge$ on $U_1^\wedge$.

$$\begin{array}{ccc} & & A_2 \\ & \nearrow \tilde{\tau} & \uparrow i_2 \\ U_1^\wedge & \xrightarrow{\psi^\wedge} & U_2^\wedge \\ \tilde{p}_1^\sim \downarrow & & \downarrow \tilde{p}_2^\sim \\ U_1 & \xrightarrow{\psi} & U_2 \end{array}$$

Then we can show that $\psi^\wedge: U_1^\wedge \rightarrow U_2^\wedge$ is continuous. To show this, it suffices to say that $\psi^\wedge$ is continuous at any $a \in A_1$, and hence, for any sequence $\{a_n\} \subset U_1^\wedge - A_1$ which converges to a , $\{\psi^\wedge(a_n)\}$ converges and $\lim_{n \rightarrow \infty} \psi^\wedge(a_n) = \psi^\wedge(a)$.

$\{\psi^\wedge(a_n)\} = \{\tilde{p}_2^{\sim -1}(\psi \circ \tilde{p}_1^\sim(a_n))\} \subset U_2^\wedge - A_2$ has cluster points in $U_2^\wedge$ since $\tilde{p}_2^\sim$ is proper, and they must be contained in A_2 . Further, the cluster points are unique and coincide with $\psi^\wedge(a)$. In fact, if α is a cluster point of $\{\psi^\wedge(a_n)\}$, we have a subsequence $\{a'_n\}$ of $\{a_n\}$ with $\lim_{n \rightarrow \infty} \tilde{p}_2^{\sim -1} \circ \psi \circ \tilde{p}_1^\sim(a'_n) = \alpha$. Then

$$\begin{aligned} \alpha &= r_2(\alpha) = r_2\left(\lim_{n \rightarrow \infty} \tilde{p}_2^{\sim -1} \circ \psi \circ \tilde{p}_1^\sim(a'_n)\right) \\ &= \lim_{n \rightarrow \infty} r_2 \circ \tilde{p}_2^{\sim -1} \circ \psi \circ \tilde{p}_1^\sim(a'_n) = \lim_{n \rightarrow \infty} \tau(a'_n) \\ &= \lim_{n \rightarrow \infty} \tilde{\tau}(a'_n) = \tilde{\tau}(a) = \psi^\wedge(a). \end{aligned}$$

Hence $\lim_{n \rightarrow \infty} \psi^\wedge(a_n) = \psi^\wedge(a)$. Therefore $\psi^\wedge$ is continuous. Since $U_k^\wedge$ is a complex manifold ($k = 1, 2$) and $\psi^\wedge|(U_1^\wedge - A_1)$ is holomorphic on $U_1^\wedge - A_1$, $\psi^\wedge$ is holomorphic on $U_1^\wedge$. Further, $\psi \circ \tilde{p}_1^\sim = \tilde{p}_2^\sim \circ \psi^\wedge$ on $U_1^\wedge$.

To complete the proof of the theorem, it suffices to show that $\psi^\wedge$ is bijective and its inverse is holomorphic. By (*), we also have the holomorphic mapping $(\psi^{-1})^\wedge: U_2^\wedge \rightarrow U_1^\wedge$ such that $\psi^{-1} \circ \tilde{p}_2^\sim = \tilde{p}_1^\sim \circ (\psi^{-1})^\wedge$ on $U_2^\wedge$. Then it follows that

$$(\psi^{-1})^{\wedge} \circ \psi^{\wedge} = \text{id} \quad \text{on } U_1^{\wedge},$$

$$\psi^{\wedge} \circ (\psi^{-1})^{\wedge} = \text{id} \quad \text{on } U_2^{\wedge}.$$

Hence $\psi^{\wedge}: U_1^{\wedge} \rightarrow U_2^{\wedge}$ is biholomorphic and, in particular, $\psi^{\wedge}(A_1) = A_2$. Therefore A_k , X_k and A_k/R_k ($k = 1, 2$) are analytically equivalent respectively, and the two diagrams are analytically equivalent. Q.E.D.

REMARK 3. H. Kerner [8] has treated the case that $r_k: X_k \rightarrow A_k$ ($k = 1, 2$) is a weakly negative vector bundle and $R_k(a) = A_k$ for any $a \in A_k$.

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